PHYS 102 Midterm Exam 2 Solutions 16.04.2022

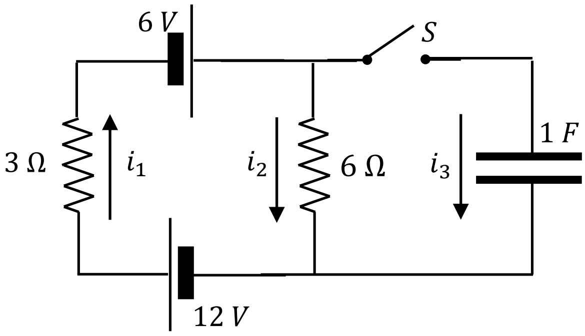
1. Consider the circuit shown in the figure with the capacitor initially uncharged. The switch S is closed at time $t = 0$.

(a) (5 Pts.) Find the currents i_1, i_2 and i_3 before the switch is closed.

(b) (5 Pts.) Find the currents i_1, i_2 and i_3 in the limit $t \to \infty$.

(c) (5 Pts.) What is the charge on the capacitor when it is full?

(d) (10 Pts.) Find the current $i_2(t)$ for $t > 0$.



Solution:

(a) Before the switch is closed $i_1 = i_2$, and $i_3 = 0$. The circuit has one loop only. Writing the loop equation, we get

$$12 - 3i_1 + 6 - 6i_1 = 0 \quad \to \quad i_1 = i_2 = 2\ \text{A}\,, \qquad i_3 = 0\,.$$

(b) In the limit $t \to \infty$ the capacitor is fully charged and no current passes through it. Hence, the result is same as in part (a).

$$i_1 = i_2 = 2\ \text{A}\,, \qquad i_3 = 0\,.$$

(c) When the capacitor is fully charged and there is no current passing through it, the potential difference on the capacitor is same as the potential difference on the 6 Ω resistor. Therefore,

$$V_C = (6\ \Omega)\ (2\ A) = 12\ V\,, \qquad V_C = \frac{Q}{C} \quad \to \quad Q = (12\ V)(1\ F) = 12\ C\,.$$

(d) After the switch is closed, we have a two-loop circuit. Junction rule gives $i_1 = i_2 + i_3$. Loop rule applied to the left loop in the clockwise direction gives

$$12 - 3i_1 + 6 - 6i_2 = 0 \quad \to \quad i_1 + 2i_2 = 6\,.$$

Loop rule applied to the right loop in the clockwise direction gives

$$6i_2 - V_C = 0 \quad \to \quad i_2 = \frac{q}{6C} = \frac{q}{6}\,,$$

where we have used the fact that the voltage on the capacitor at any time is $V_C = q/C = q$, for $C = 1\ F$. The capacitor being initially uncharged means the current i_3 is charging it. so, we have

$$i_3 = \frac{dq}{dt}\,, \qquad i_1 = 6 - 2i_2 = 6 - \frac{q}{3}\,, \qquad i_2 = \frac{q}{6}\,.$$

Using these results in the junction rule, we obtain

$$6 - \frac{q}{3} = \frac{q}{6} + \frac{dq}{dt} \quad \to \quad \frac{dq}{dt} = 6 - \frac{1}{2}q \quad \to \quad \frac{dq}{q - 12} = -\frac{dt}{2} \quad \to \quad q(t) = 12\left(1 - e^{-t/2}\right) C\,.$$

Therefore,

$$i_2 = \frac{q}{6} = 2\left(1 - e^{-t/2}\right) A\,.$$

2. A particle with charge q and mass m, initially moving along the positive x axis enters a region $0 < x < \ell$ where a uniform magnetic field $\vec{\mathbf{B}} = B_0\,\hat{\mathbf{k}}$ exists, as shown in the figure. The particle is deflected a distance d in the $+y$ direction as it traverses the field. Answer the following questions in terms of q, m, B_0, ℓ, and d.

(a) (4 Pts.) Is q positive or negative? (State your reason to get points.)

(b) (7 Pts.) What is the speed of the particle as it leaves the magnetic field?

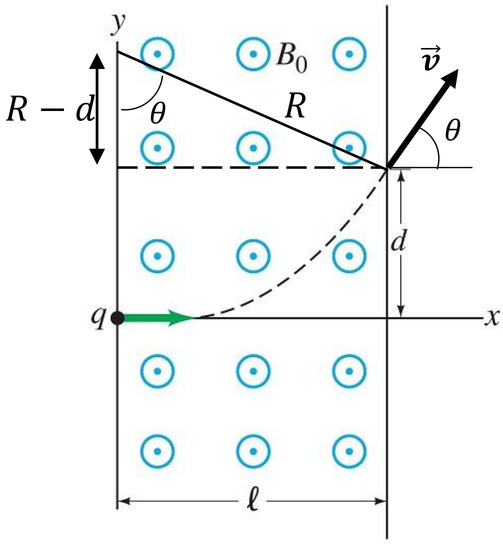
(c) (7 Pts.) What is the angle θ between particle's velocity vector and the x-axis as the particle leaves the magnetic field?

(d) (7 Pts.) What is the time it takes for the particle to traverse the magnetic field?

Solution: (a) The charge q must be negative because

$$\vec{\mathbf{v}}_0 = v_0\hat{\mathbf{i}}\,, \qquad \vec{\mathbf{B}} = B_0\hat{\mathbf{k}} \quad \rightarrow \quad \vec{\mathbf{F}}_B = q\vec{\mathbf{v}}_0 \times \vec{\mathbf{B}} = qv_0B_0(\hat{\mathbf{i}} \times \hat{\mathbf{k}}) = qv_0B_0(-\hat{\mathbf{j}})$$

and it is deflected in the $+y$-direction.



(b) Path of the charged particle in the magnetic field is an arc of a circle with radius R.

$$|q|vB_0 = m\frac{v^2}{R} \quad \rightarrow \quad v = \frac{|q|B_0}{m}R\,.$$

From the figure, we have

$$(R-d)^2 + \ell^2 = R^2 \quad \rightarrow \quad R = \frac{\ell^2 + d^2}{2d}\,.$$

Hence

$$v = \frac{|q|B_0(\ell^2 + d^2)}{2md}\,.$$

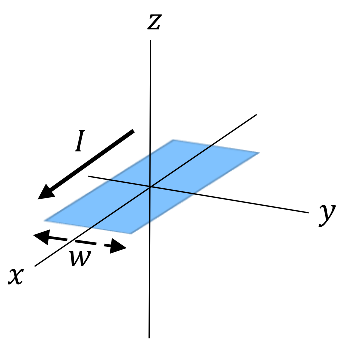
(c)

$$\sin\theta = \frac{\ell}{R} = \frac{2\ell d}{\ell^2 + d^2} \quad \rightarrow \quad \theta = \arcsin\left(\frac{2\ell d}{\ell^2 + d^2}\right) = \arctan\left(\frac{2\ell d}{\ell^2 - d^2}\right) = \arccos\left(\frac{\ell^2 - d^2}{\ell^2 + d^2}\right).$$

(d) Period of the circular motion is $T = 2\pi R/v$. The time t it takes for the particle to traverse the magnetic field is found as

$$t = \frac{\theta}{2\pi}T \quad \rightarrow \quad t = \frac{m}{|q|B_0}\arctan\left(\frac{2\ell d}{\ell^2 - d^2}\right).$$

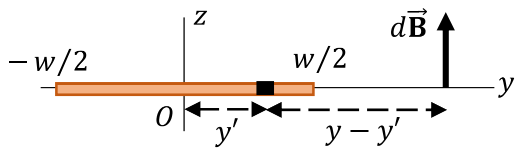
3. (25 Pts.) A very long (consider as ∞), thin conducting sheet of width w placed on the xy-plane is oriented along the x-axis ($-w/2 < y < w/2$). It is carrying a current I in the $+x$-direction. Find the magnetic field $\vec{\mathbf{B}}(y)$ created by this current on the y-axis for $y > w/2$.



Solution: Magnetic field created by an infinite line of current is found by using Ampère's law where the path of integration is a circle of radius r centred at the current.

$$\oint \vec{\mathbf{B}} \cdot d\vec{\boldsymbol{\ell}} = B(2\pi r) = \mu_0 I \quad \rightarrow \quad B = \frac{\mu_0 I}{2\pi r}.$$

We divide the sheet carrying the current into infinitesimal strips with width dy', at a distance y' from the origin as shown. Current in each infinitesimal strip is $dI = (I/w)dy'$. Magnetic field created by the infinitesimal strip is

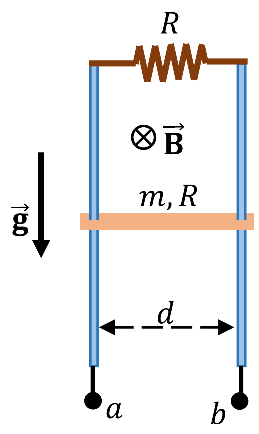


$$d\vec{\mathbf{B}} = \frac{\mu_0 dI\, \hat{\mathbf{k}}}{2\pi(y - y')} = \frac{\mu_0 I}{2\pi w}\left(\frac{dy'}{y - y'}\right)\hat{\mathbf{k}}.$$

Magnetic field created by the sheet is found by integration.

$$\vec{\mathbf{B}} = \frac{\mu_0 I\, \hat{\mathbf{k}}}{2\pi w}\int_{-w/2}^{w/2} \frac{dy'}{y - y'} \quad \rightarrow \quad \vec{\mathbf{B}} = \frac{\mu_0 I}{2\pi w} \ln\left|\frac{y + w/2}{y - w/2}\right| \hat{\mathbf{k}}.$$

4. A horizontal conducting bar of mass m and resistance R which is initially at rest, starts sliding down two parallel, vertical, frictionless, and conducting rails always maintaining contact. The rails have negligible resistance, and are connected to each other at the top by a resistance R. A uniform magnetic field of magnitude B exists in the region between the rails perpendicular to and directed into the plane of the paper shown in the figure.



(a) (5 Pts.) What is the terminal speed of the bar?

(b) (10 Pts.) Find the speed of the bar as a function of time.

(c) (10 Pts.) If a battery is to be connected to the terminals a and b to stop the bar from falling down, what should be its emf and polarity (a negative, b positive or, a positive, b negative? Explain your answer.)

Solution: As the bar slides down the emf induced around the loop is $\mathcal{E}_{\text{ind}} = Bdv(t)$, where $v(t)$ is the speed of the bar at time t. The current induced around the loop is $I_{\text{ind}} = \mathcal{E}_{\text{ind}}/2R$. The net force acting on the bar is the force of gravity pulling it down, and the magnetic force $F_B = I_{\text{ind}}Bd$ which is opposing the motion (Lenz's law). Therefore,

$$mg - F_B = m\frac{dv}{dt} \quad \rightarrow \quad \frac{dv}{dt} = g - \frac{B^2d^2}{2mR}v \quad \rightarrow \quad \frac{dv}{dt} = -\frac{B^2d^2}{2mR}\left(v - \frac{2mgR}{B^2d^2}\right).$$

(a) Terminal speed is reached when the bar no longer accelerates. Therefore,

$$\frac{dv}{dt} = 0 \quad \rightarrow \quad v_T - \frac{2mgR}{B^2d^2} = 0 \quad \rightarrow \quad v_T = \frac{2mgR}{B^2d^2}.$$

(b) Rearranging the differential equation, and integrating we get

$$\frac{dv}{dt} = -\frac{B^2d^2}{2mR}(v - v_T) \quad \rightarrow \quad \frac{dv}{v - v_T} = -\frac{B^2d^2}{2mR}dt \quad \rightarrow \quad v(t) = v_T\left(1 - e^{-\frac{B^2d^2t}{2mR}}\right).$$

(c) To stop the bar from falling down, we connect a battery with emf $\mathcal{E}$ to terminals a and b. This will cause a current flow $I = \mathcal{E}/R$ through the bar which will cause a magnetic force $F = IBd = \mathcal{E}Bd/R$ acting on the bar. The net force on the bar must be zero. Therefore,

$$F = \frac{\mathcal{E}Bd}{R} = mg \quad \rightarrow \quad \mathcal{E} = \frac{mgR}{Bd}.$$

Positive terminal of the battery should be connected to terminal a so that a counter clockwise current results in an upward magnetic force to balance the downward gravitational force.